

How to use the code?

(version 11.11.17)

You only need to work on 2 main files

- `main.m` : change all coefficients and play with models.
- `getLambda.m` : change the form of penalty coefficient λ , cf. section 3

The results are given in the folder `results/`

- `errors.txt` : the mesh size (`h`) w.r.t errors $\|e\|_{L^2}$ (denoted by `L2` in the code), $\|k\nabla e\|_{L^2}$ (`L2G`), $\|e\|_{H^1}$ (`H1`) and the energy norm (`ENorm`, cf. section 3.2 in Barrau's thesis [1]).
- `errPlot.jpg`: graph of the convergence of errors in L^2 and $ENorm$.
- `onlyMesh.jpg` : figure of the mesh.
- `exactSolution.jpg` : figure of the exact solution.
- `numericalSolution.jpg` : figure of the numerical solution.

The questions related to the idea, the algorithms about the implementation, please read on the file `nx fem_matlab_algorithm.pdf`.

1 File main.m

You can change the models (cf. section 2), values of all parameters in the test. Note that, read the comment next to each code line for more information about the name of each parameter.

Step to `%INPUT` region

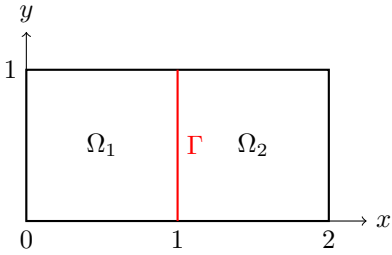
- `mdl` : choose the model (1 for Sinha's, 2 and 3 for Barrau's), cf. section 2 for more details about these models.
- `findCR` : you wanna see the behavior of convergence rate (CR)?
 - 1 if you wanna see the convergence rate (CR) as well as the error in some norms. The CR will display on the terminal, the errors will be given in a data file whose name is `errors.txt`. If you wanna change the number of segment on each iteration, you can change the expression of `numSeg(z)` in the section `% MAIN PROCESS` (in the middle of the file `main.m`).
 - 0 if you just wanna see the plot of solution or the error $u - u_{ex}$. The figures are given in the folder `results`. Please go to section `%PLOTING` (at the end of file `main.m`) and change the value which are described in the code.
- `pa.lambdaH` : the value of $\hat{\lambda}$. Please see section 3 for more detail on the form of λ . Note that, this $\hat{\lambda}$ doesn't depend on the mesh size or the position of the interface.
- `pa.kap1per2` : set to 1 if you want $\kappa_1 = \kappa_2 = \frac{1}{2}$ instead of the more-complicated form suggested in Becker's paper [2]. Keep 0 for otherwise.
- `pa.useGP` : for the moment, we keep 0 because we don't work on the ghost penalty terms. We also don't care about `pa.gam1`, `pa.gam2`.
- `pa.reguMesh` : 1 if you wanna use the regular mesh, 0 for otherwise.
- `nSeg` : number of segment, just to be used in the case `findCR=0`.
- In the section `% settings for each model`

- `pa.kk1`, `pa.kk2` : value of diffusion coefficients. Note that, for the Sinha’s test case (`mdl=1`), `pa.kk1` must be double `pa.kk2`. Other test cases, you can use whatever values you want.
- `pa.xi` : ξ indicating the interface, please see more at section 2. Note that, for the Sinha’s test case, the interfae is fixed to $\xi = 1$.

2 Models

There are 3 available models you can work with,

■ **Sinha** (cf. [4]). All information about this model are given in the file `sinha.m`. This model is very simple, the boundary condition is clear ($u = 0$ on $\partial\Omega$). The limit of this test case is that the exact solution doesn’t depend on the diffusion coefficients and the difference between these coefficients is not much.

$$\begin{aligned}
 -\nabla \cdot (k\nabla u) &= f, & \text{in } \Omega, \\
 \llbracket u \rrbracket &= 0, & \text{on } \Gamma, \\
 \llbracket k\nabla_{\mathbf{n}} u \rrbracket &= 0, & \text{on } \Gamma, \\
 u &= 0, & \text{on } \partial\Omega.
 \end{aligned} \tag{1}$$


Where, $\Omega = [0, 2] \times [0, 1]$, $\Gamma = \{1\} \times [0, 1]$, $k_1 = 1, k_2 = \frac{1}{2}$ (in general, $k_1 = 2k_2$).

Exact solution (doesn’t belong to k_1, k_2),

$$u_{\text{ex}}(x, y) = \begin{cases} u_1(x, y) = \sin(\pi x) \sin(\pi y) \\ u_2(x, y) = -\sin(2\pi x) \sin(\pi y) \end{cases}$$

function `defExSol`.

The right hand side :

$$f = \begin{cases} f_1 = 2k_1\pi^2 \sin(\pi x) \sin(\pi y) \\ f_2 = -5k_2\pi^2 \sin(2\pi x) \sin(\pi y) \end{cases}$$

function `defF`.

In Sinha’s model (1), domain is a rectangular $[0; 2] \times [0; 1]$.

■ **Barrau with x** (cf. [1], page 43). All information about this model are given in the file `barrau.m`. With this model, we can test the dependency of the solution w.r.t the diffusion coefficients. It’s limit is that the interface is only a straight line ($x = \xi$).

Domain $\Omega = [-1, 1] \times [-1, 1]$, $\Gamma = \{0.91\} \times [-1, 1]$. Parameters $k_1 = 1, k_2 = 1000$.

Exact solution

$$\begin{aligned}
 -\nabla \cdot (k\nabla u) &= -4, & \text{in } \Omega, \\
 \llbracket u \rrbracket &= 0, & \text{on } \Gamma, \\
 \llbracket k\nabla_{\mathbf{n}} u \rrbracket &= 0, & \text{on } \Gamma, \\
 u &= u_{\text{ex}}, & \text{on } \partial\Omega.
 \end{aligned} \tag{2}$$

$$u_{\text{ex}} = \begin{cases} \frac{x^2}{k_1} (x \leq \xi) \\ \frac{x^2 - \xi^2}{k_2} + \frac{\xi^2}{k_1} (x > \xi). \end{cases}$$

■ **Barrau with a part of circle** (cf. [1], page 47). All information about this model are given in the file `barrau_r.m`. The interface in this case is a part of a circle.

Domain $\Omega = [0, 1] \times [0, 1]$, $r_0 = 0.75$. Parameters $k_1 = 1, k_2 = 1000$.

Exact solution

$$\begin{aligned}
 -\nabla \cdot (k \nabla u) &= -2, & \text{in } \Omega, \\
 \llbracket u \rrbracket &= 0, & \text{on } \Gamma, \\
 \llbracket k \nabla_{\mathbf{n}} \rrbracket &= 0, & \text{on } \Gamma, \\
 u &= u_{ex}, & \text{on } \partial\Omega.
 \end{aligned}
 \quad
 u_{ex} = \begin{cases} \frac{r^2}{k_1} (r \leq r_0) \\ \frac{r^2 - r_0^2}{k_2} + \frac{r_0^2}{k_1} (r > r_0). \end{cases}$$

with $r = \sqrt{x^2 + y^2}$.

■ **Barrau with a full circle in center** (cf. [1], page 113)

3 Form of λ (file `getLambda.m`)

There are several choices of form of λ described in the file `getLambda.m`.

- `%% just constant` : $k_1 = k_2 = \frac{1}{2}$.
- `%% Barrau's thesis p.31`:

$$\hat{\lambda} \max\{k_1, k_2\} \frac{1}{h_T}, \quad \text{on cut-triangles } T \text{ only.}$$

- `%% Becker's article` :

$$\hat{\lambda} \max \left\{ k_1 \frac{|K_1|}{|K|}, k_2 \frac{|K_2|}{|K|} \right\} \frac{|\Gamma_h|}{|K|}$$

- `%% Barrau's thesis p.32` (recommended by Barrau) :

$$\hat{\lambda} \frac{k_1 k_2 |\Gamma_h|}{k_2 |K_1| + k_1 |K_2|}$$

- `%% Annavarapu's article` (recommended by Burman [3]) :

$$\hat{\lambda} \frac{h_T}{h} \frac{k_1 k_2 |\Gamma_h|}{k_2 |K_1| + k_1 |K_2|}.$$

References

- [1] Nelly Barrau. “Generalisation de la méthode Nitsche XFEM pour la discretisation de problèmes d’interface elliptiques”. PhD thesis. Université de Pau et des Pays de l’Adour, 2013, p. 197.
- [2] Nelly Barrau et al. “A robust variant of NXFEM for the interface problem”. In: *Comptes Rendus Mathématique* 350.15-16 (Aug. 2012), pp. 789–792. ISSN: 1631073X. DOI: [10.1016/j.crma.2012.09.018](https://doi.org/10.1016/j.crma.2012.09.018).
- [3] Erik Burman et al. “CutFEM: Discretizing geometry and partial differential equations”. In: *INTERNATIONAL JOURNAL FOR NUMERICAL METHODS IN ENGINEERING Int.* (2014). ISSN: 0743-1619. DOI: [10.1002/nme](https://doi.org/10.1002/nme).
- [4] Rajen Kumar Sinha and Bhupen Deka. “An unfitted finite-element method for elliptic and parabolic interface problems”. eng. In: *IMA journal of numerical analysis* 27.3 (2005), pp. 529–549. ISSN: 1464-3642.